Lie Algebra

define a Lie algebra as an $\overset{(\mathbb{R},\mathbb{C})}{F}$-vector space $\mathfrak{g}$ with a bilinear operation:

$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ s.t. · alternating $[x,x]=0$

· Jacobi id. : $[x,[y,z]]$ + cyclic $= 0$

$\to [x,y] = -[y,x]$

Associative algebras, $[x,y] = xy - yx$

last week: matrix lie groups from $GL(n)$

lie algebras: $\mathfrak{gl}(n)$ $n \times n$ matrices n^2-dim'l

$\mathfrak{sl}(n)$ $\quad \mathrm{tr}\,\xi = 0$ $\quad n^2-1$

$\mathfrak{so}(n)$ $\quad \xi^T = -\xi$ $\quad \frac{1}{2}n(n-1)$

$\mathfrak{su}(n)$ $\quad \xi^\dagger = -\xi,\ \mathrm{tr}\,\xi = 0$ $\quad n^2-1$

$\mathfrak{so}(n)$: spanned by n^2-1 basis 'vectors'

$n=2$: $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$, $\xi, \eta \in \mathfrak{so}(2)$ then $[\xi,\eta] = \lambda\lambda'\left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right] = 0$

$n=3$: $\Bigg\{\underbrace{\begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}}_{L_1}, \underbrace{\begin{pmatrix} 0 & & 1 \\ & 0 & \\ -1 & & 0 \end{pmatrix}}_{L_2}, \underbrace{\begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}}_{L_3}\Bigg\}$, $[L_i, L_j] = \epsilon_{ijk} L_k$

Exponential Map

Want to connect G with $\mathfrak{g}$ (T_eG) by defining $\exp : \mathfrak{g} \to G$

consider a curve $\gamma : \mathbb{R} \to G$ satisfying $\gamma(s) \cdot \gamma(t) = \gamma(s+t)$

then, $\gamma(0) = e$ and $\gamma'(0) = \xi$ $\quad \left(\gamma'(0) = \frac{d\gamma^\mu(0)}{dt}\partial_\mu\right)$

together, this uniquely defines γ, "one-parameter subgroup"

$\partial_t \gamma(s+t)|_{t=0} = \gamma'(s) = \gamma(s)\cdot\gamma'(0) = \gamma(s)\cdot\xi$

so, let $\exp(\xi) = \gamma(1)$ for γ s.t. $\gamma'(0) = \xi$

remark: $\exp(r\xi) = \gamma(r)$

example (subgroups of) $GL(n)$: $\gamma'(t) = \gamma(t)\xi$

$\Rightarrow \gamma(t) = e^{t\xi}$ & $\exp(\xi) = e^{\xi}$

matrix exponential: $e^{\xi} \equiv \sum_{k=0}^{\infty} \frac{\xi^k}{k!}$

• $\mathfrak{so}(2) \to SO(2)$: $\exp\Big(t\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{(\ )^2 = -\mathbb{1}}\Big) = \sum_{k\,\text{even}} \frac{t^k(-1)^{\frac{k}{2}}}{k!}\mathbb{1} + \sum_{k\,\text{odd}} \frac{t^k(-1)^{\frac{k-1}{2}}}{k!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, 2D rotation matrix

• $\mathfrak{so}(3) \to SO(3)$: $\xi = rL_1 + sL_2 + tL_3$

$\exp(\xi)$ is 3D rotation matrix

note: · $(\mathbb{R}^*, \times)$: $\gamma'(t) = \gamma(t)\xi = e^{\xi} \in \mathbb{R}_{>0}$

· $(\mathbb{R}, +)$: $\gamma'(t) = \partial_s(\gamma(s) + \gamma(t))|_{s=0} = \gamma'(0) = \xi$

$\exp(\xi) = \xi \in \mathbb{R}$

* exp is not generally surjective

exp connects lie algebra to lie group

$\exp(x)\exp(y) \neq \exp(x+y)$ in general

instead $= \exp(x + y + [x,y] + \ldots)$ (BCH)

$\to$ failure to commute is translated

Riemannian Geometry

$X, Y \in \mathfrak{X}(M)$ smooth sections $M \to TM$, $X(p) \in T_pM$

locally, $X = a^\mu\partial_\mu$, $Y = b^\mu\partial_\mu$, $a^\mu, b^\nu \in C^\infty(U)$

lie bracket $[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ s.t. $[X,Y] = XY - YX$

for $f \in C^\infty(M)$, $[X,Y](f) = X(Yf) - Y(Xf)$

$\overset{\text{loc.}}{=} (a^\mu\partial_\mu b^\nu - b^\mu\partial_\mu a^\nu)\partial_\nu f$

here bracket defined on all of M, not just e

instead, identify $\xi \in T_eM$ with left-invariant vector fields $X^\xi \in \mathfrak{X}^L(M)$

where $X^\xi(g) = L_{g*}\xi$

if M is a matrix lie group, $X^\xi(g) = g\cdot\xi$

exponential map, $\exp_p : T_pM \to M$

one-parameter subgrp $\rightsquigarrow$ geodesic γ s.t. $\gamma(0) = p$ & $\gamma'(0) = \xi$

$\exp_p(t\xi) = \gamma(t)$

Extra

Show G is a manifold, and what is its algebra

· $SL(n) = \{A \in M_n(\mathbb{R}) : \det A = 1\}$

differential $d\det_A B = \lim_{h\to 0} \frac{\det(A+hB) - \det A}{h}$

$= \det A\ \mathrm{tr}(A^{-1}B)$

for $A \in \det^{-1}(1)$, if $B = \frac{\lambda}{n}A$, $d\det_A(B) = \lambda$

$\therefore$ surjective $\underset{\text{(pre-image thrm)}}{\Rightarrow}$ $SL(n)$ is an n^2-1 dim. manifold

$T_A SL(n) = \{B \in M_n(\mathbb{R}) : \mathrm{tr}(A^{-1}B) = 0\}$ so,

$\mathfrak{g} = \{$traceless $n \times n$ matrices$\}$